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Investigation of the Problem of Singularities in General Relativity

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Recently Addy and Datta have obtained a linearized solution for isentropic motions of a perfect fluid by assigning Cauchy data on the hypersurface $x^4=0$ and by imposing a restriction on the equation of state. In the present paper we pursue this study and discuss the problem of singularities from the standpoint of a local observer for which a singularity is defined as a state with an infinite proper rest mass density. It is shown that for a closed universe with any distribution of matter whatsoever there occurred a singularity in the past in the nonrotating parts of the universe and it must recur in the future. Furthermore, the collapse of a rotating fluid to a singularity seems inevitable when the relativistic equation of state is considered.

1. INTRODUCTION

Pachner (1968, 1971) has studied the isentropic motions of a perfect fluid by using comoving coordinates in the framework of general relativity and has investigated the problem of singularities without assuming any symmetry in the line element. Bera and Datta (1974, 1975), Datta (1975–1976, 1976–1977, 1977), and Basu, Dalal, and Datta (to be published) have pursued this study and obtained linearized solutions by dealing with the Cauchy problem. Recently Addy and Datta (to be

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published) have obtained a linearized solution for isentropic motions of a perfect fluid by assigning Cauchy data on the hypersurface $x^4=0$ and by imposing a restriction on the equation of state. In the present paper we pursue this study and discuss the problem of singularities from the standpoint of a local observer for which a singularity is defined as a state with an infinite proper rest mass density. It is shown that for a closed universe with any distribution of matter whatsoever there occurred a singularity in the past in the nonrotating parts of the universe and it must recur in the future. Furthermore, the collapse of a rotating fluid to a singularity seems inevitable when the relativistic equation of state is considered. This is in complete agreement with the conclusion of Hawking and Ellis (1968, 1973).

2. THE LINEARIZED SOLUTION OF A PERFECT FLUID

In brief we write out the linearized solution for isentropic motions of a perfect fluid obtained very recently by Addy and Datta (to be published) by assigning Cauchy data g_{ik}, $g_{ik,4}$ on the hypersurface $x^4=0$:

$$ds^2=g_{\mu\nu}dx^\mu dx^\nu, \qquad \text{signature } (+++-) \tag{2.1}$$

where

$$g_{\mu\nu}=g_{\mu\nu}(x^1,x^2,x^3,x^4) \tag{2.2}$$

$$g_{i4}=u_i=0 \qquad (i=1,2) \tag{2.3}$$

$$g_{34}=u_3/\mu=c_3(x^2,x^3)/\mu^2 \tag{2.4}$$

$$g_{44}=u_4/\mu=-\frac{1}{\mu^2} \tag{2.5}$$

$$\mu=1+\epsilon+p/\rho \tag{2.6}$$

where p is the pressure, ρ the proper rest mass density, ϵ the proper internal energy per unit mass, and u^ν the timelike 4-velocity. In the comoving reference frame defined by

$$u^i=\delta^i_4 \qquad (i=1,2,3) \tag{2.7}$$

one obtains

$$u^4=(-g_{44})^{-1/2} \tag{2.8}$$

and the conservation law of baryon number is reduced to

$$\rho=(g_{44}/g)^{1/2}f(x^i) \tag{2.9}$$

where $f(x^i)$ is a function of space coordinates and may be determined by the initial distribution of matter. The energy momentum tensor for a perfect fluid is defined by

$$T_\mu^\nu=\rho\mu u_\mu u^\nu+p\delta_\mu^\nu \tag{2.10}$$

We consider that all thermodynamical processes are adiabatic and assume that μ is a function of x^4 alone. The Cauchy data $g_{ik}, g_{ik,4}$ on the hypersurface $x^4=0$ are chosen to satisfy

$$(g_{ik})_0=(1+2\lambda)\delta_{ik}, \qquad |\lambda|\ll 1 \tag{2.11}$$

$$(g_{ik,4})_0=0 \qquad (i,k=1,2,3) \tag{2.12}$$

where

$$\lambda=\frac{1}{b}\int\left[\rho\frac{(1+\epsilon)}{r}\right]dv \qquad (b=\text{const}) \tag{2.13}$$

$$dv=dx^1dx^2dx^3 \tag{2.14}$$

r is the spatial distance (in Euclidean metric) of dv from the point at which λ is computed and the integration is taken over the hypersurface $x^4=0$. Thus in terms of Cauchy data given by (2.11)–(2.14) and the derivatives $(g_{ik,44})_0$ of the Cauchy data explicitly given by the independent field equations

$$R_{ik}=-8\pi\left(T_{ik}-\tfrac{1}{2}Tg_{ik}\right) \tag{2.15}$$

the linearized solution of Addy and Datta (personal communication) can be expressed by the power series

$$g_{ik}=(g_{ik})_0+\tfrac{1}{2}(x^4)^2(g_{ik,44})_0+\cdots \tag{2.16}$$

3. INVESTIGATION OF SINGULARITIES

In order to deal with the problem of singularities from the standpoint of a local observer one must know the past and future of the proper rest mass density of the perfect fluid whose behavior is determined by the

Einstein field equations and as such one must express the relation (2.9) in a differential form. Differentiating (2.9) with respect to x^4, one obtains

$$\rho_{,4}/\rho = \tfrac{1}{2}\left(g_{44,4}/g_{44} - g^{\mu\nu} g_{\mu\nu,4}\right) \tag{3.1}$$

In view of the equation

$$R_{44} = -8\pi\left(T_{44} - \tfrac{1}{2} T g_{44}\right) \tag{3.2}$$

one may obtain following Pachner (1971)

$$\begin{aligned}\frac{\rho_{,44}}{\rho} = {} & \frac{4\pi\rho(\mu + 2p/\rho)}{\mu^2} + \frac{4}{3}\left(\frac{\rho_{,4}}{\rho}\right)^2 \\ & + \left[\chi^{\alpha\beta}\chi_{\alpha\beta} - \frac{1}{3}\left(\frac{\rho_{,4}}{\rho}\right)^2 - \left(\frac{\chi_{44}}{g_{44}}\right)^2\right] - \omega^{\alpha\beta}\omega_{\alpha\beta} \\ & + \left[\frac{1}{2}\frac{g_{44,44}}{g_{44}} - g^{\alpha\beta}\left(\omega_{\alpha 4} + \chi_{\alpha 4}\right)_{;\beta}\right]\end{aligned} \tag{3.3}$$

where

$$\chi_{\alpha\beta} = \tfrac{1}{2} g_{\alpha\beta,4} \tag{3.4}$$

$$\omega_{\alpha\beta} = \tfrac{1}{2}\left(g_{\alpha 4,\beta} - g_{\beta 4,\alpha}\right) \tag{3.5}$$

Following Pachner (1971), Bera and Datta (1975), and Datta (1975–1976, 1976–1977, 1977), we define Φ^2 by

$$\Phi^2 = \tfrac{1}{4}\gamma^{ia}\gamma^{kb}\gamma_{ik,4}\gamma_{ab,4} - \tfrac{1}{12}\left(\gamma^{ik}\gamma_{ik,4}\right)^2 \tag{3.6}$$

where γ_{ik}, γ^{ik} are connected to the metric tensor components $g_{\mu\nu}$ by the relations (Landau and Lifshitz, 1962)

$$\gamma_{ik} = g_{ik} - g_{i4}g_{k4}/g_{44} \tag{3.7}$$

$$\gamma^{ik} = g^{ik} \tag{3.8}$$

The geometrical properties of space are defined by the positive definite metric

$$d\sigma^2 = \gamma_{ik}\, dx^i dx^k \tag{3.9}$$

Furthermore, the quantity Φ^2 expresses the influence of the shear and remains invariant with respect to coordinate transformations which are restricted by the coordinate conditions (2.5) and (2.7). Thus one may choose at any given moment a system of coordinates in which equations (2.3)–(2.5) hold and whose spatial axes are at a particular point orthogonal to each other. If the motion is isotropic at the point, the invariant Φ^2 vanishes in this system and is given by

$$\Phi^2 = \tfrac{1}{4} g^{ia} g^{kb} g_{ik,4} g_{ab,4} - \tfrac{1}{12}\left(g^{ik} g_{ik,4}\right)^2 + \tfrac{1}{6}\left(g^{34} g_{34,4}\right)^2 + \tfrac{1}{2} g^{33} g^{34} g_{33,4} g_{34,4} - \tfrac{1}{6} g^{ik} g^{34} g_{ik,4} g_{34,4} \tag{3.10}$$

In view of (3.1), (3.4), and (3.10) one may get

$$\chi^{\alpha\beta}\chi_{\alpha\beta} - \tfrac{1}{3}(\rho_{,4}/\rho)^2 - 2(\chi_{44}/g_{44})^2 = \Phi^2 - \tfrac{1}{4}(g_{44,4}/g_{44})^2 \tag{3.11}$$

The only nonvanishing components of $\omega_{\alpha\beta}$ are given by

$$\omega_{23} = -c_{3,2}/2\mu^2 = -\omega_{32} \tag{3.12}$$

$$\omega_{34} = -c_3\mu_{,4}/\mu^3 = -\omega_{43}$$

In view of (2.6), (3.12) and the conservation law of energy

$$\frac{d\epsilon}{d\rho} = \frac{p}{\rho^2} \tag{3.13}$$

one may get

$$\left(\frac{dp}{d\rho}\right)^2 g^{\mu\nu} Q_\mu Q_\nu = \rho^2 g^{33} (c_3\mu_{,4})^2 \tag{3.14}$$

where

$$Q_1 = \rho_{,1}, \qquad Q_2 = \rho_{,2}, \qquad Q_3 = \rho_{,3} + c_3\rho_{,4}, \qquad Q_4 = 0 \tag{3.15}$$

Next, from (3.12) and (3.14), we get

$$\omega^{\alpha\beta}\omega_{\alpha\beta} = \frac{2|\Omega|^2}{\mu^2} - \frac{2}{\mu^4\rho^2}\left(\frac{dp}{d\rho}\right)^2 g^{\mu\nu} Q_\mu Q_\nu \tag{3.16}$$

where the square of the angular velocity is given by (Gödel, 1949; Taub, 1956)

$$|\Omega|^2 = g_{\mu\nu}\Omega^\mu\Omega^\nu = g^{33}(c_{3,2})^2/4\mu^2 g_{22} \tag{3.17}$$

$$\Omega^\nu = \tfrac{1}{2}(-g)^{-1/2}\epsilon^{\nu\alpha\beta\gamma}u_\alpha u_{\beta,\gamma}$$

As the components $\omega_{\mu 4}$ and $\chi_{\mu 4}$ behave like the components of four-dimensional covariant vectors under the coordinate transformations permitted by the coordinate conditions (2.5) and (2.7), one may calculate

$$\frac{1}{2}\frac{g_{44,44}}{g_{44}} - g^{\alpha\beta}(\omega_{\alpha 4} + \chi_{\alpha 4})_{;\beta}$$

$$= \frac{1}{2}\left(\frac{g_{44,4}}{g_{44}}\right)^2 + (-g)^{-1/2}\left[g^{\alpha\beta}Q_\beta \frac{dp}{d\rho}(-g)^{1/2}/\mu^3\rho\right]_{,\alpha} \tag{3.18}$$

In view of (3.1), (3.11), (3.16), and (3.18), and using the relation

$$ds = (-g_{44})^{1/2}dx^4 = dx^4/\mu \tag{3.19}$$

one may write the differential relation (3.3) in the form

$$\frac{1}{\rho}\frac{\partial^2\rho}{\partial s^2} = 4\pi\rho\left(\mu + \frac{2p}{\rho}\right) + \left(\frac{2}{\mu^2\rho^2}\right)\left(\frac{dp}{d\rho}\right)^2 g^{\mu\nu}Q_\mu Q_\nu$$

$$+\left[\frac{4}{3} + \left(\frac{1}{\mu}\frac{dp}{d\rho}\right)^2 + \frac{1}{\mu}\frac{dp}{d\rho}\right]\left(\frac{1}{\rho}\frac{\partial\rho}{\partial s}\right)^2 + \mu^2\Phi^2$$

$$+\mu^2(-g)^{-1/2}\left[g^{\alpha\beta}Q_\beta\frac{dp}{d\rho}(-g)^{1/2}/\mu^3\rho\right]_{,\alpha} - 2|\Omega|^2 \tag{3.20}$$

For nonrotating matter in equilibrium we must have

$$\frac{\partial^2\rho}{\partial s^2} = \frac{\partial\rho}{\partial s} = |\Omega|^2 = \Phi = 0 \tag{3.21}$$

and all derivatives with respect to x^4 or s must vanish. Thus the condition for equilibrium of a perfect fluid comes out from (3.20):

$$4\pi\rho\left(\mu+\frac{2p}{\rho}\right)+\left(\frac{2}{\mu^2\rho^2}\right)\left(\frac{dp}{d\rho}\right)^2(g^{ik}\rho_{,i}\rho_{,k})(-g)^{1/2}$$
$$=-\left[g^{ik}\rho_{,k}\left(\frac{1}{\rho}\frac{dp}{d\rho}\right)(-g)^{1/2}/\mu\right]_{,i} \tag{3.22}$$

Thus one may conclude that

$$-\left[g^{ik}\rho_{,k}\left(\frac{1}{\rho}\frac{dp}{d\rho}\right)(-g)^{1/2}/\mu\right]_{,i} \tag{3.23}$$

represents the elastic forces which counterbalance the gravitational attraction created by the rest mass, the pressure, and the square of the pressure gradient. Consequently equation (3.22) is the relativistic analog of the classical equation of the hydrostatic equilibrium

$$4\pi\rho G=-\operatorname{div}\left(\frac{1}{\rho}\operatorname{grad}p\right) \tag{3.24}$$

where G is the Newtonian constant of gravitation.

Now one can deal with the problem of singularities by examining the differential relation (3.20). This relation gives Raychaudhuri's formula (1955) for noncoherent matter with cosmological constant $\Lambda=0$:

$$\rho_{,44}/\rho=4\pi\rho+\tfrac{4}{3}(\rho_{,4}/\rho)^2+\Phi^2-2|\Omega|^2 \tag{3.25}$$

where Φ^2 assumes the form

$$\Phi^2=\tfrac{1}{4}g^{ia}g^{kb}g_{ik,4}g_{ab,4}-\tfrac{1}{12}\left(g^{ik}g_{ik,4}\right)^2 \tag{3.26}$$

In the case of irrotational motion one may have

$$c_3=0 \tag{3.27}$$

by a suitable coordinate transformation subject to the coordinate conditions (2.5) and (2.7) and the equation (3.20) can be reduced to

$$\frac{1}{\rho}\frac{\partial^2\rho}{\partial s^2}=4\pi\rho\left(1+\epsilon+\frac{3p}{\rho}\right)+\frac{2}{\mu^2\rho^2}\left(\frac{dp}{d\rho}\right)^2(g^{ik}\rho_{,i}\rho_{,k})$$
$$+\left[\frac{4}{3}+\left(\frac{1}{\mu}\frac{dp}{d\rho}\right)^2+\frac{1}{\mu}\frac{dp}{d\rho}\right]\left(\frac{1}{\rho}\frac{\partial\rho}{\partial s}\right)^2+\mu^2\Phi^2$$
$$+\mu^2(-g)^{-1/2}\left[g^{\alpha k}\rho_{,k}\left(\frac{1}{\rho}\frac{dp}{d\rho}\right)(-g)^{1/2}\Big/\mu^3\right]_{,\alpha} \qquad (3.28)$$

It is evident from (3.22) that if the mass of the nonrotating fluid in stable equilibrium exceeds a certain critical limit and if the equation of state is changed owing to the nuclear burning so that the elastic forces (3.23) cannot counterbalance the gravitational attraction created by the rest mass, the pressure and the square of the pressure gradient, the fluid starts contracting. As contraction proceeds, the increasing pressure in the first term of the right-hand member of (3.28), the kinetic energy of the fluid in the third term, the anisotropy in the isentropic motion in the fourth term, and the deviation from the uniform distribution of matter in the second term accelerate the gravitational collapse to a singularity which is reached in a finite interval of proper time (Oppenheimer and Volkoff, 1939; Landau 1932).

This conclusion, drawn from the standpoint of a local observer, can be applied to cosmology and one may state that for a closed universe with any distribution of matter whatsoever there occurred a singularity in the past in the nonrotating parts of the universe and it must recur in future (Hawking and Ellis, 1968, 1973; Bera and Datta, 1975; Datta, 1975–1976, 1976–1977, 1977).

We note in passing that the conclusion that the deviation from a spherically symmetric distribution of a nonrotating fluid as represented by the square of the pressure gradient in the second term of the right-hand member of (3.28) accelerates the gravitational collapse is in agreement with that of Penrose (1965).

As the possibility of rotation of the universe in the large is admitted (Wolfe, 1970), we investigate the problem of gravitational collapse for rotational motion.

In view of (2.5), (2.9), and (3.17), one may write

$$|\Omega|^2=g_{11}(c_{3,2}/f)^2(\rho/2\mu)^2 \qquad (3.29)$$

From (3.29) it follows that for highly condensed matter the effect of rotation which opposes contraction is of the order of ρ^2, while that of attraction in the first term of the second member of (3.20) or (3.25) is of the order of ρ. Thus in spite of the fact that the gravitational attraction together with the square of the pressure gradient, the kinetic energy, and the anisotropy in the motion of a rotating fluid supports contraction, it appears from equation (3.20) that the rotation together with the elastic forces may possibly lead to the condition

$$\frac{\partial^2 \rho}{\partial s^2} < 0 \tag{3.30}$$

during the contraction of space. Hence it seems that rotational motion may likely stop contraction and avoid the occurrence of a singularity (Pachner, 1971; Bera and Datta, 1975; Datta 1975–1976, 1976–1977, 1977). Likewise, from equation (3.25) corresponding to noncoherent matter one may conclude that a universe filled with spinning dust may not possibly contract to a singularity. This is in agreement with and supplemented by the conclusions of Trautman (1972a, 1972b, 1973), Kopczyński (1972, 1973), and Tafel (1973), who have studied the problems on the basis of the Einstein–Cartan theory of gravitation.

If during the contraction of a rotating fluid, its proper rest mass density attains very high magnitude and the equation of state becomes relativistic, then

$$p = \rho(1+\epsilon)/3 \tag{3.31}$$

Substituting the value of p from (3.31) in (3.13) and integrating, one may get

$$1+\epsilon = (\rho/\rho_0)^{1/3} \tag{3.32}$$

$$p/\rho = \tfrac{1}{3}(\rho/\rho_0)^{1/3} \tag{3.33}$$

where ρ_0 denotes the proper rest mass density with zero internal energy. From (3.32) it follows that

$$\epsilon < 0, \qquad \text{when } \rho < \rho_0 \tag{3.34}$$

Hence the relativistic equation of state (3.31) may be applicable only for $\rho \gg \rho_0$, for which case ρ_0 may be treated as a constant.

In the problem of gravitational collapse of a rotating fluid the first term of the right-hand member of (3.20) representing the gravitational attraction and the last term representing the rotational motion play the dominant role when the equation of state becomes relativistic. In the case under consideration we note that in view of (3.29) and (3.33) these two terms assume the values

$$4\pi\rho(\mu+2p/\rho)=8\pi\rho_0^{-1/3}\rho^{4/3} \tag{3.35}$$

$$|\Omega|^2=g_{11}\rho^{4/3}\left(3c_{3,2}\rho_0^{1/3}/8f\right)^2 \tag{3.36}$$

Comparing these results with the previous ones one may notice that the high pressure increases the influence of gravitational attraction from the order ρ to $\rho^{4/3}$, while it diminishes the influence of rotation from the order ρ^2 to $\rho^{4/3}$. But as the kinetic energy which becomes relativistic during the contraction of space gives strong support to contraction, one may conclude that in spite of the fact that the rotation (3.36) together with the elastic forces opposes contraction, the gravitational attraction together with other terms supporting contraction may lead to the condition

$$\frac{\partial^2\rho}{\partial s^2}>0 \tag{3.37}$$

Thus the collapse of a rotating fluid to a singularity is inevitable when the relativistic equation of state is considered.

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